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# FIXED POINTS OF COUNTABLY CONDENSING MULTIMAPS HAVING CONVEX VALUES ON QUASI-CONVEX SETS

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Abstract. We obtain a Chandrabhan type fixed point theorem for a multimap having a non-compact domain and a weakly closed graph, and taking convex values only on a quasi-convex subset of Hausdorff locally convex topological vector space. We introduce the definition of Chandrabhan-set and find a sufficient condition for every countably condensing multimap to have a relatively compact Chandrabhan-set. Finally, we establish a new version of Sadovskii fixed point theorem for multimaps.

## 1. Introduction and preliminaries

In 1967, Sadovskii [19] defined the condensing single-valued function and proved that a condensing function from a closed bounded convex subset of a Banach space into itself has a fixed point. Daher [7] generalized the concept of the condensing function to countably condensing functions, which is condensing only on countable sets.

Mönch [14] introduced a new class of single-valued functions, later called a Mönch type function by Dhage [9] and he proved a fixed-point theorem for it. Mönch [14], Mönch and von Harten [15], Deimling [8], Guo et al. [10], Agarwal and O'Regan [1] and O'Regan and Precup [17] obtained fixed point theorems for Mönch type operators and applied them to differential and integral equations. A Mönch type multimaps was relaxed to Chandrabhan multimaps by Dhage [9].

A multimap (or simply, a map) $F : X \multimap Y$ is a function from a set $X$ into the power set of $Y$. Throughout this paper, we assume that maps have nonempty values otherwise explicitly stated or obvious from the

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context. We abbreviate a Hausdorff locally convex topological vector space as HLCTVS.

The following fixed point theorem is stated in Cardinali and Papalini [4]:

Theorem 1.1. *Let $E$ be a HLCTVS, $K$ be a nonempty compact subset of $E$ and $G : K \multimap K$ be a map taking closed values and with the properties*

(1) *there exists a quasi-convex subset $A$ of $K$ such that $\overline{A} = K$ and $G(x)$ is convex for every $x \in A$; and*
(2) *$G$ has a weakly closed graph.*

*Under these conditions, there exists an $x \in K$ such that $x \in G(x)$.*

Cardinali, O'Regan and Rubbioni [3] defined a Mönch-set for a multimap defined on HLCTVS and got a Mönch type fixed point theorem whose Mönch hypothesis is weaker than those of [5], [6], [17].

In Section 2, we extend Theorem 1.1 to a new fixed point theorem for multimaps defined on non-compact subsets of HLCTVS. Motivated by [3], we introduce the definition of a Chandrabhan-set for a multimap and verify that the sufficient conditions for the existence of the Mönch-set and the Chandrabhan-set are the same. We obtain a Chandrabhan type fixed point theorem for a map having a non-compact domain and a weakly closed graph, and taking convex values only on a quasi-convex subset of HLCTVS. This result generalizes those of [3], [5], [6], [13], [17].

In Section 3, we find conditions for that every countably condensing map has a relatively compact Chandrabhan-set if the domain of the map is a subset of a HLCTVS. In this case, the HLCTVS satisfies the Krein-Smulian property and its compact subsets are separable. Finally, we establish a new version of Sadovskii fixed point theorem for maps in HLCTVS only with the Krein-Smulian property.

Definition 1.2. *A nonempty subset $Y$ of a HLCTVS $E$ is said to be quasi-convex (or almost convex) if for any $V \in \mathcal{V}$, where $\mathcal{V}$ is a neighborhood system of the origin $0$ in $E$, and for any finite set $\{y_1, y_2, ..., y_\delta\} \subset Y$, there exists a finite set $\{z_1, z_2, ..., z_n\} \subset Y$ such that $z_i - y_i \in \mathcal{V}$ for each $i = 1, 2, ..., n$ and $co\{z_1, z_2, ..., z_n\} \subset Y$.*

For example, deleting a certain subset of the boundary of a closed convex set, we get a quasi-convex set. For details, see [11, 18].

Definition 1.3. ([4, 5].) *Let $X$ be a nonempty subset of a HLCTVS $E$. It is said that a map $G : X \multimap E$ has a weakly closed graph in*

$X \times E$ *if for every net*$(x_\delta)_\delta$ *in* $X$, $x_\delta \to x$, $x \in X$, *and for every net* $(y_\delta)_\delta$, $y_\delta \in G(x_\delta)$, $y_\delta \to y$, *then* $S(x,y) \cap G(x) \neq \emptyset$, *where* $S(x,y) = \{x + \lambda(y-x) : \lambda \in [0,1]\}$.

## 2. Chandrabhan-sets and fixed point theorems

LEMMA 2.1. *Let* $E$ *be a HLCTVS,* $X$ *be a closed convex subset of* $E$, $B$ *be a relatively compact subset of* $X$ *and* $F : X \multimap X$ *be a map. Then there exists a subset* $K$ *of* $X$ *such that* $K = co(B \cup F(K))$.

*Proof.* Put $K_0 = \mathrm{co}(B)$, $K_{n+1} = \mathrm{co}(B \cup F(K_n))$ for $n = 0, 1, 2, \cdots$ and $K = \bigcup_{n=0}^{\infty} K_n$. By induction, $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \cdots$. Note that $K$ is convex, since $K_n$ is convex for $n = 0, 1, 2, \cdots$.

Now we can show that $K = \mathrm{co}(B \cup F(K))$. For each $n$, $\mathrm{co}(B \cup F(K_n)) \subseteq \mathrm{co}(B \cup F(K))$, so $K = \bigcup_{n=0}^{\infty} \mathrm{co}(B \cup F(K_n)) \subseteq \mathrm{co}(B \cup F(K))$. On the other hand, $K$ is a convex set containing $B$ and $\bigcup_{n=0}^{\infty} F(K_n) = F(K)$, hence $\mathrm{co}(B \cup F(K)) \subseteq K$.

□

We extend Theorem 1.1 with the following new fixed-point theorem for multimaps defined on noncompact domains:

THEOREM 2.2. *Let* $E$ *be a HLCTVS,* $X$ *be a closed convex subset of* $E$ *and* $Y$ *be a subset of* $X$ *such that* $K \cap Y$ *is quasi-convex and* $\overline{K \cap Y} = \overline{K}$ *for any relatively compact convex subset* $K$ *of* $X$. *Assume that* $F : X \multimap X$ *is a map with closed values satisfying the followings:*

(1) $F(x)$ *is convex for every* $x \in Y$;
(2) $F$ *has a weakly closed graph; and*
(3) *there exists a relatively compact subset* $B$ *such that* $K = co(B \cup F(K))$ *is relatively compact.*

*Then* $F$ *has a fixed point.*

*Proof.* Consider the map $T : \overline{K} \multimap \overline{K}$ defined by $T(x) = F(x) \cap \overline{K}$ for all $x \in \overline{K}$, where the set $K = \mathrm{co}(B \cup F(K))$ is found in Lemma 2.1. Then the map $T$ has nonempty values. In fact, fixed $x \in \overline{K}$, there exists a net $(x_\delta)_\delta$ in $K$ such that $x_\delta \to x$. Let us consider a net $(y_\delta)_\delta$ such that $y_\delta \in F(x_\delta)$. Since $F(K) \subset K$ and $\overline{K}$ is compact, there is an $y \in \overline{K}$ such that $y_\delta \to y$. By (2), $S(x,y) \cap F(x) \neq \emptyset$. As the convexity of $\overline{K}$ implies $S(x,y) \subset \overline{K}$, $T(x) = F(x) \cap \overline{K} \neq \emptyset$.

The above discussion also shows that $\emptyset \neq S(x,y) \cap F(x) = S(x,y) \cap F(x) \cap \overline{K} = S(x,y) \cap T(x)$, so $T$ has a weakly closed graph in $\overline{K} \times \overline{K}$.

Furthermore $Y \cap \overline{K}$ is dense in $\overline{K}$. As $F$ takes closed values in $X$ and convex values in $Y$, $T$ satisfies all the assumptions of Theorem 1.1. Therefore, there exists $x \in \overline{K}$ such that $x \in T(x) \subset F(x)$. □

Definition 2.3. *Let $X$ be a convex subset of a HLCTVS $E$, $B$ be a relatively compact subset of $X$ and $F : X \multimap X$ be a given map. We say that a set $A \subset X$ a Chandrabhan-set for $F$ if $A = \mathrm{co}(B \cup F(A))$ and there exists a countable subset $C$ of $A$ with $\overline{A} = \overline{C}$.*

*When $B = \{x_0\}$ for some $x_0 \in X$, $A$ is called a Mönch-set for $F$ in* [3].

Consider a HLCTVS $E$ satisfying the following properties:

$(X1)$ If $A$ is a compact subset of $E$, then $\overline{\mathrm{co}}(A)$ is compact.

$(X2)$ For any relatively compact subset A of X, there exists a countable set $B \subset A$ such that $\overline{B} = \overline{A}$.

If $E$ is a quasi-complete HLCTVS, then (X1) holds. (X1) is called the Krein-Smulian property. If $E$ is metrizable, then (X2) holds. For details, see [3, 16].

Lemma 2.4. *Let $E$ be a HLCTVS satisfying (X1) and (X2), $X$ be a closed convex subset of $E$ and $B$ be a relatively compact subset of $X$. Suppose that a multimap $F : X \multimap X$ maps compact sets into relatively compact sets. Then $F$ has a Chandrabhan-set.*

*Proof.* As the proof of Lemma 2.1, put $K_0 = \mathrm{co}(B)$, $K_{n+1} = \mathrm{co}(B \cup F(K_n))$ for $n = 0, 1, 2, \cdots$ and $K = \bigcup_{n=0}^{\infty} K_n$, then $K = \mathrm{co}(B \cup F(K))$.

Let us prove by induction that $K_n$ is relatively compact for $n = 0, 1, 2, \cdots$. Assumption (X1) implies that $K_0$ is relatively compact and so is $K_1$. Suppose that $K_n$ is relatively compact for $n \geq 2$. Because $\overline{K_{n+1}} \subset \overline{\mathrm{co}}(\overline{B} \cup \overline{F(\overline{K_n})})$ and $F$ maps compact sets into relatively compact sets, $K_{n+1}$ is relatively compact.

Now, we verify that $K$ is a Chandrabhan-set $K$ for $F$. By (X2), there exists a countable subset $C_n$ of $K_n$ such that $\overline{C_n} = \overline{K_n}$ for $n = 0, 1, 2, \cdots$. Put $C = \bigcup_{n=0}^{\infty} C_n$, then $\overline{C} = \overline{K}$, since $\overline{K} = \overline{\bigcup_{n=0}^{\infty} K_n} = \bigcup_{n=0}^{\infty} \overline{K_n} = \bigcup_{n=0}^{\infty} \overline{C_n} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C}$. □

Using Lemma 2.4, we obtain the following Chandrabhan type fixed point theorem, which specifies the conditions in Theorem 2.2:

Theorem 2.5. *Let $E$ be a HLCTVS satisfying (X1) and (X2), $X$ be a closed convex subset of $E$ and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset*

*$K$ of $X$. Assume that $F : X \multimap X$ is a map with closed values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*
(2) *$F$ has a weakly closed graph;*
(3) *$F$ maps compact sets into relatively compact sets; and*
(C) *there exists a relatively compact subset $B$ such that a Chandrabhan-set for $F$ is relatively compact.*

*Then $F$ has a fixed point.*

Corollary 2.6. *Let $E$ be a HLCTVS satisfying (X1) and (X2). Let $X$ be a closed convex subset of $E$ and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset $K$ of $X$. Assume that $F : X \multimap X$ is a map with closed values satisfying conditions (1), (2) and (3) in Theorem 2.5 and the following:*

(M) *there exists an $x_0 \in X$ such that a Mönch-set for $F$ is relatively compact.*

*Then $F$ has a fixed point.*

For $X = Y$ and $F$ has a compact values, Corollary 2.6 reduces to Theorem 5.2 in [3]. Cardinali et al. [3] improved all the theorems in the literature (see, e.g. Theorem 3.1 in [5], Theorem 3.1 in [6]) by assuming (M) instead of the following condition:

(M1) There exists an $x_0 \in X$ such that every Mönch-set for $F$ is relatively compact.

Since separable Banach spaces endowed with the weak topology satisfy (X1) and (X2), we obtain the following corollary from Theorem 2.5:

Corollary 2.7. *Let $X$ be a closed convex subset of a separable Banach space $E$ endowed with the weak topology $\mathcal{T}_w$, and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y}^w = \overline{K}^w$ for any relatively $w$-compact convex subset $K$ of $X$. Assume that $F : X \multimap X$ is a map with closed values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*
(2) *$F$ has a $w$-weakly closed graph;*
(3) *$F$ maps $w$-compact sets into relatively $w$-compact sets; and*
(C) *there exists a relatively $w$-compact subset $B$ such that a Chandrabhan-set for $F$ is relatively $w$-compact.*

*Then $F$ has a fixed point.*

Remark 2.8. (1) Note that $\overline{K}^w$ is the weak closure of $K$. If $\overline{K}^w$ is weakly compact ($w$-compact, for short), the set $K$ is said to be relatively $w$-compact. It is said that $F$ has a $w$-weakly closed graph in $X \times X$ if it has weakly closed graph in $X \times X$ with respect to $\mathcal{T}_w$.

(2) If $X = Y$, $B = \{x_0\}$ and assuming (M1) instead of (C), Corollary 2.7 becomes Theorem 3.1 [6].

Since Banach spaces satisfy (X1) and (X2), we get the following corollary which generalizes Theorem 2.1 in [13].

Corollary 2.9. *Let $X$ be a closed convex subset of a Banach space $E$, and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset $K$ of $X$. Assume that $F : X \multimap X$ is a map with compact values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*
(2) *$F$ has a weakly closed graph;*
(3) *$F$ maps compact sets into relatively compact sets; and*
(C) *there exists a relatively compact subset $B$ such that a Chandrabhan-set for $F$ is relatively compact.*

*Then $F$ has a fixed point.*

## 3. Fixed point theorems for countably condensing maps

Definition 3.1. *Let $E$ be a HLCTVS satisfying (X1), $\mathcal{P}_b(E) = \{H \subset E : H \neq \emptyset, H \text{ bounded}\}$. A function $\beta : \mathcal{P}_b(E) \multimap \mathbb{R}_0^+$ is called a* measure of noncompactness *(MNC, for short) on $E$ provided that the following conditions hold for any $A, B \in \mathcal{P}_b(E)$:*

(1) *$\beta(\overline{\text{co}}A) = \beta(A)$; and*
(2) *$\overline{A}$ is compact iff $\beta(A) = 0$.*

*A set additive MNC $\beta$ is an MNC $\beta$ that satisfies the following condition:*

(3) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$.

For details, see [2, 12]. Clearly a set additive MNC $\beta$ satisfies the properties

(4) monotonicity: $A \subset B$ implies $\beta(A) \leq \beta(B)$; and
(5) nonsingularity: $\beta(A \cup \{x\}) = \beta(A)$ for every $x \in E$.

Definition 3.2. *Let $X$ be a nonempty subset of a HLCTVS $E$ satisfying (X1) and let $\beta$ be a MNC. A map $F : X \multimap E$ is said to be (countably) condensing if*

(I) *$F(X)$ is bounded; and*

(II) *$\beta(F(B)) < \beta(B)$ for all (countable) bounded subsets $B$ of $X$ with $\beta(B) > 0$.*

The condition (II) can be equivalently formulated as

(II′) for all (countable) bounded subsets $B$ of $X$, the relation $\beta(B) \leq \beta(F(B))$ implies that $\overline{B}$ is compact.

From now on, we only consider a countably condensing map defined with respect to a set additive MNC.

LEMMA 3.3. *Let $X$ be a closed convex subset of a HLCTVS $E$ satisfying (X1) and (X2). Suppose that a countably condensing map $F : X \multimap X$ maps compact sets into relatively compact sets. Then every Chandrabhan-set for $F$ is relatively compact.*

*Proof.* Let $B$ be a relatively compact subset of $X$ and $A$ be a Chandrabhan-set for $F$ according to Lemma 2.4, that is, $A = \mathrm{co}(B \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset $C$ of $A$.

Every point of $C$ can be written as a finite combination of points belonging to the set $B \cup F(A)$, so there exists a countable set $M \subset A$ such that $C \subset \mathrm{co}(B \cup F(M))$. By the definition of a countably condensing map, $F(X)$ is bounded, and the sets $A$, $C$ and $M$ are also bounded. Since $\beta(B) = 0$,

(*) $$\beta(C) \leq \beta(\mathrm{co}(B \cup F(M))) = \beta(B \cup F(M)) = \beta(F(M)).$$

Let us show that $\beta(M) = 0$. If not, then $\beta(F(M)) < \beta(M)$, because $F$ is countably condensing. Combining above argument, we obtain

$$\beta(C) \leq \beta(F(M)) < \beta(M) \leq \beta(A) = \beta(\overline{A}) = \beta(\overline{C}) = \beta(C),$$

a contradiction. Therefore $\overline{M}$ is compact.

Now, we prove $\beta(\overline{A}) = 0$. As $F$ maps compact sets into relatively compact sets, $\beta(\overline{F(\overline{M})}) = 0$. Hence $\beta(F(M)) = 0$ and $\beta(C) = 0$ by (*), which implies that $\beta(\overline{A}) = \beta(C) = 0$, that is, $\overline{A}$ is compact. □

By Lemma 3.3 and Theorem 2.5, we obtain the following theorem:

THEOREM 3.4. *Let $E$ be a HLCTVS satisfying (X1) and (X2), $X$ be a closed convex subset of $E$ and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset $K$ of $X$. Assume that $F : X \multimap X$ is a countably condensing map with closed values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*

(2) *$F$ has a weakly closed graph; and*

(3) *$F$ maps compact sets into relatively compact sets.*
*Then $F$ has a fixed point.*

Remark 3.5. (1) If $E$ is a Banach space, Theorem 3.4 reduces to Theorem 3.4 in [13].

(2) For $X = Y$ and $F$ has a compact convex values, Theorem 3.4 is Theorem 5.4 in [3]. A special case of Theorem 5.4 in [3] is Theorem 4.1 in [6] where $E$ is a separable Banach space endowed with the weak topology $\mathcal{T}_w$.

## 4. Sadovskii type theorem

Without assuming neither that $E$ satisfies (X2) nor that the map $F$ maps compact sets into relatively compact sets, we obtain a following fixed point theorem for condensing maps defined with respect to a nonsingular MNC:

Theorem 4.1. *Let $E$ be a HLCTVS satisfying (X1), $X$ be a closed convex subset of $E$ and $Y$ be a subset of $X$ such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset $K$ of $X$. Assume that $F : X \multimap X$ has a closed valued condensing map with respect to a nonsingular MNC and satisfies the followings:*

(1) *$F(x)$ is convex for every $x \in Y$; and*
(2) *$F$ has a weakly closed graph.*

*Then $F$ has a fixed point.*

*Proof.* For $x_0 \in X$, consider the family $\{H_\alpha\}_\alpha$ of all subsets of $E$ that each satisfies the following properties:

(i) $x_0 \in H_\alpha$;
(ii) $H_\alpha$ is closed and convex; and
(iii) $F(X \cap H_\alpha) \subset H_\alpha$.

Put $H = \bigcap_\alpha H_\alpha$, then $H$ is well-defined, since $X \in \{H_\alpha\}_\alpha$.

Let us prove that $H \in \{H_\alpha\}_\alpha$. Clearly, $H$ satisfies (i) and (ii). Moreover, since $F(X \cap H) \subset F(X \cap H_\alpha) \subset H_\alpha$ for all $\alpha$, $H$ satisfies (iii).

Now, to prove $\overline{\text{co}}(\{x_0\} \cup F(H)) = H$, let us first verify $\overline{\text{co}}(\{x_0\} \cup F(H)) \subset H$. As $X \in \{H_\alpha\}_\alpha$, $H \subset X$ and using property of (iii) of $H$, we obtain $F(H) = F(X \cap H) \subset H$. Because $H$ satisfies (i) and (ii),

$$(**) \qquad \overline{\text{co}}(\{x_0\} \cup F(H)) \subset H.$$

To verify that $H \subset \overline{\text{co}}(\{x_0\} \cup F(H))$, it is enough to show $\overline{\text{co}}(\{x_0\} \cup F(H)) \in \{H_\alpha\}_\alpha$. The set $\overline{\text{co}}(\{x_0\} \cup F(H))$ satisfies (i) and (ii) and by (**), $F(X \cap \overline{\text{co}}(\{x_0\} \cup F(H))) \subset F(X \cap H) = F(H) \subset \overline{\text{co}}(\{x_0\} \cup F(H))$.

Finally, we will show that $H$ is compact. Because $F(X)$ is bounded and $H \subset X$, so is $F(H)$. Therefore $H = \overline{\text{co}}(\{x_0\} \cup F(H))$ is bounded. Suppose that $\beta(H) > 0$, then

$$\beta(F(H)) < \beta(H) = \beta(\overline{\text{co}}(\{x_0\} \cup F(H))) = \beta(F(H))$$

which is a contradiction. Therefore $\beta(H) = 0$ and the closed set H is compact.

As $F|_H$ satisfies all the hypotheses of Theorem 1.1, there exists $x \in H$ such that $x \in F(x)$.

□

REMARK 4.2. (1) For $X = Y$, Theorem 4.1 becomes Theorem 5.4 in [3]. Theorem 4.1 in [6], where $X = Y$ and $E$ is a separable Banach space endowed with the weak topology $\mathcal{T}_w$, is a special case of Theorem 4.1.

(2) The proof of Theorem 4.1 uses the idea of [3, 6], but simplifies it by removing unnecessary assumptions.

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